Feasibility Bound for the Invariant-Based Rank-1 OCRS in the Adversarial Order Model

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Abstract

In the rank-1 matroid setting with adversarial order, we prove that there is an OCRS that guarantees each active element a 1/2 probability of selection. In an OCRS, or online contention resolution scheme, items come in one by one. Each item might be *active*, and if it is, the chooser must decide right away whether to accept it or not. Once you pass, you can't go back. In the rank-1 case, you can accept only one item in total. The proof here is based on the construction in Appendix A.1 of [2], which is originally inspired by the "Magician's Problem" from Saeed Alaei [1].

Problem Setup. We begin with an ordering of items, whose order is chosen by an adversary. The order can be thought as $1, 2, ..., n$. Item i is active, and demands with probability x_i. Each item i demands independently of the others, and $\sum_i x_i \leq 1$. Since this is a rank-1 matroid, the chooser only selects one item. We show that for each i, $\Pr[i \text{ is accepted} \mid i \text{ is active}] \leq \alpha$, $\alpha \leq \frac{1}{2}$.

Theorem 1 (Feasibility Bound for Rank-1 OCRS)**.** *In the rank-1 matroid setting with adversarial order, feasibility requires $\alpha \leq \frac{1}{2}$. Together with the construction achieving $\alpha \geq \frac{1}{2}$, it follows that the algorithm is exactly a $\frac{1}{2}$-selectable OCRS.*

Proof. Lets define r_i as the probability rule for reaching item i without selecting anything beforehand. We want to ensure that $q_i r_i = \alpha$, $\forall i$, so we let α start as an arbitrary constant, with the bound that $\alpha \in [0, 1]$ By construction, if α works, if $q_i \leq 1$, $\forall i$, then α is valid, which allows each active item to have acceptance probability α.

$r_i = \Pr[\text{reach } i \text{ without having accepted anyone}]$ with $r_1 = 1$. Upon reaching i, we consider it with probability $q_i = \alpha / r_i$ so that $q_i r_i = \alpha$. If considered and active, we accept i and stop.

For each i, conditioning on i being active, $\Pr[\text{accept } i \mid i \text{ active}] = r_i \cdot \frac{\alpha}{r_i} = \alpha$.

We reach $i + 1$ if, and only if we reach i and fail to accept i. Denote x_i as the probability item i is active. The probability of failing to accept i (of course given that we accepted it) is $1 - q_i x_i$. It follows that the reach probability for the next item $(i + 1)$ is $r_{i+1} = r_i \cdot (1 - q_i x_i)$.

From the recurrence $r_{i+1} = r_i - \alpha x_i$ with $r_1 = 1$, we can show by induction that

$$r_i = 1 - \alpha \sum_{k<i} x_k.$$

Since $\sum_{k=1}^{n} x_k \leq 1$, this immediately implies $r_i \geq 1 - \alpha$ for all i. For $q_i \leq 1$ (probabilities must be in $[0, 1]$), we need $r_i \geq \alpha$, which requires, in the smallest case of r_i for $r_i = \alpha$, implying $1 - \alpha \geq \alpha \implies \alpha \leq \frac{1}{2}$. Setting $\alpha = \frac{1}{2}$ gives a feasible algorithm achieving $\Pr[\text{accept } i \mid i \text{ active}] = \frac{1}{2}$ for all i. $\square$

Conclusion. This feasibility result means that, for the algorithm in Appendix A.1 of [2], the choice of α is forced to be exactly $\frac{1}{2}$. Appendix A.1 in [2] chooses a $r_n = \alpha$, giving $\alpha \geq \frac{1}{2}$. Combined with the feasibility bound $\alpha \leq \frac{1}{2}$, this forces $\alpha = \frac{1}{2}$.

This means that we can guarantee that every active item an acceptance probability of exactly 50% in the rank-1 adversarial-order setting, and this is the largest possible value that keeps the algorithm feasible.

References

[1] Saeed Alaei. Bayesian combinatorial auctions: Expanding single buyer mechanisms to many buyers. *SIAM Journal on Computing*, 43(2):930–972, 2014.

[2] Moran Feldman, Ola Svensson, and Rico Zenklusen. Online contention resolution schemes, 2018.